B-spline Based Functional F Test
For Functional Linear Models

A thesis submitted in partial satisfaction
of the requirements for the degree Master of Science
in Statistics

by

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2008
The thesis of Yan Chen is approved.

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2008
To My Parents

And

My Love
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ABSTRACT OF THE THESIS

B-spline Based Functional F Test
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Functional linear regression model where the responses are functions and the predictors are scalar vectors is considered. B-spline based functional $F$ test (BF test) which is a hybrid of B-splines and functional $F$ test is illustrated by an application from a robust design study. Simulations are conducted to compare the powers of BF test with functional $F$ test and B-spline based multivariate likelihood ratio test (BM test) under three covariance structures. Robustness of Gaussian process is investigated as well. Results show that BF test and functional $F$ test are more reliable than BM test with certain advantages and Gaussian distribution for assumed error process is not crucial.
CHAPTER 1

Introduction

Functional data analysis becomes more and more popular as data sets are collected through a process naturally described as function. The “function” refers to the intrinsic structure of the data rather than to their explicit form. The basic philosophy of functional data analysis is that “we should think of, model and analyze the observed data functions as single entities, rather than merely a sequence of individual observations” [6].

In certain fields of study, such as physics, engineering sciences, and public health research, data are sometimes collected with high frequency over long periods of time. Additionally, there is no any obvious parametric form and the shape of the curves varies after exploring the data. Hence, it is not reasonable to make strong parametric assumptions without any parametric form when the data have a relatively large number of observations. For instance, one robust parameter design experiment (the audible noise data) reported by Nair et al. [4] which has 43 measurements of sound pressure levels were collected at rotating speeds ranging from 1000 to 2500 revolution per minute. To analyze such experimental data with large-grid responses, it might not be appropriate to apply traditional multivariate multiple regression or longitudinal data analysis. These methodologies are mainly designed for data with a relatively small number of measures.
per individual and high variation in the measurement. The individual’s progress or curve can only reasonably be approximated by a simple parametric form, often linear, because there are insufficient data to try more complex models. Even though there are some advanced models such as nonlinear mixed effects models can be applied on such functional data set, the computation cost is considerable and the results are hard to interpret; or some may involve parameters with unverifiable assumptions. Therefore functional data analysis is utilized to address the problems caused by data collected with high dimensionality.

Functional data analysis is a collection of strategies for analyzing functional data sets, such as curves, images, or shapes. Much of the prior work has been done. Ramsay and Dalzell [5] provided some general ideas on functional data analysis. Ramsay and Silverman [6] provided a comprehensive introduction to functional data analysis. Fan and Lin [1] developed a basis function type approach for functional ANOVA models. The functional data analysis fills a gap in current methodology to better handle cases in which many observations per unit are recorded. And it can be further used as a precursor to parametric repeated measures models. Most importantly, functional data analysis could draw more robust conclusions since it has features similar to nonparametric methods, requiring fewer assumptions on the intra-subject error correlation and mean structures for the studied population [7]. Functional data analysis has essentially the same goals as those of any other branch of statistics. It includes the following:

- To represent the data in ways that help for further statistical analysis
- To display the data so as to highlight various characteristics
➢ To study important sources of pattern and variation among the data

➢ To explain variation in an outcome or dependent variable by using input or independent variable information

➢ To compare two or more sets of data with respect to certain types of variation, where two sets of data can contain different sets of replicates of the same functions, or different functions for a common set of replicates.

Functional regression analysis is an emerging statistical approach where either the response or the predictors or both could be functions. In this thesis, we consider a functional linear regression model where the responses are functions and the predictors are scalar vectors. Faraway [2] pointed out the inappropriateness of traditional multivariate test statistics and proposed predictive models for functional responses. Shen and Faraway [7] proposed a functional F test and did simulations to compare the functional F test with the multivariate likelihood ratio test and the B-spline based test; they found that functional F test had at least the following advantages:

➢ It works when the grid size becomes large;

➢ It is stable and not easily influenced by unimportant variation directions;

➢ It is computationally cheap.

Meanwhile, functional F test has the disadvantage: it might not be the most powerful test. B-spline based multivariate likelihood ratio test is a powerful test under certain situations; however it might produce statistically significant, but practically less meaningful, test results. Shen and Xu [8] further provided diagnostics for linear models with functional

An objective of this thesis is to propose a valuable approach, $B$-spline based functional $F$ test (or BF test) in order to have a more powerful test and alleviate the difficulty of the choice of the number of basis functions. We perform simulations to compare three methods, namely, functional $F$ test (or F-test), BF test, and $B$-spline based multivariate likelihood ratio test (or BM test). We further investigate whether F-test and BF test are robust to the assumption of Gaussian process. This thesis is organized as follows. Various models and methods of analysis are laid out in Chapter 2. A practical example from a robust design study is illustrated in Chapter 3. In the following Chapter 4, simulations are performed to compare three tests. Conclusions and discussions are given in Chapter 5.
CHAPTER 2

Functional Linear Regression Models

2.1 Functional linear regression analysis

Consider the situation where the \( i \)-th response is a smooth real function \( y_i(t) \), \( i=1, \ldots, n, t \in \tau \). Responses are considered to be functions because this is a better and easier way to handle data collected continuously over a period of time; of course in reality, it is only possible to observe the function \( y_i(t) \) at a finite number of points.

Consider a functional linear regression model where the responses are functions and the predictors are scalar vectors. The model has the familiar form of

\[
y_i(t) = x_i^T \beta(t) + \varepsilon_i(t)
\]

where \( x_i = (x_{i1}, \ldots, x_{ip})^T \) is a vector of predictor variables, \( \beta(t) = (\beta_1(t), \ldots, \beta_p(t))^T \) is a vector of coefficient functions, and \( \varepsilon_i(t) \) is an error function of Gaussian stochastic process with mean zero and a covariance function \( \gamma(s,t) = \text{cov}(\varepsilon_i(s), \varepsilon_i(t)) \). It is also assumed that \( \varepsilon_i(t) \) and \( \varepsilon_j(t) \) are independent of each other for \( i \neq j \) and it is allowed for correlated errors within individuals but not between individuals.
Since in practice, the functional data are collected at a finite number of points, the least squares estimator which is a non-smoothed, unbiased estimator of $\beta(t)$ is recommended [2] to estimate the unknown coefficient function $\beta(t)$ and the form is

$$\hat{\beta}(t) = (X^T X)^{-1} X^T Y(t)$$

(2)

where $X = (x_1, \ldots, x_n)^T$ is the usual $n \times p$ model matrix and $Y(t) = (y_1(t), \ldots, y_n(t))^T$ is the vector of response functions. The predicted (or fitted) responses are $\hat{y}_i(t) = x_i^T \hat{\beta}(t)$ and the residuals are $\hat{\epsilon}_i(t) = y_i(t) - \hat{y}_i(t)$ . The residual sum of squares is

$$rss = \sum_{i=1}^{n} \| \hat{\epsilon}_i \|^2 = \sum_{i=1}^{n} \int \hat{\epsilon}_i(t)^2 dt .$$

Consider the comparison of two nested linear models, $\omega$ and $\Omega$ , where $\text{dim} (\Omega) = p$ and $\text{dim} (\omega) = q$ . The model $\omega$ results from a linear restriction on the parameters of model $\Omega$ . There are relatively few satisfactory solutions available in the statistical literature to this situation. A naïve approach is to examine the point-wise $t$ statistics on each grid point for testing $\beta(t)$ , this method carries on a serious problem with multiple-comparison and if Bonferroni correction were applied to the significance level, power would be significantly compromised because the responses are often highly correlated within each individual. As pointed out by Faraway [2], traditional multivariate test statistics such as Wilks’ lambda likelihood ratio are inappropriate due to the influence of unimportant variation directions.
2.2 Functional F test for hypothesis testing (F-test)

To overcome issues noted in section 2.1, Shen and Faraway [7] proposed a functional F test (or F-test), in order to solve these issues. Define

\[
F = \frac{(rss_\omega - rss_\Omega)/(p - q)}{rss_\Omega/(n - p)}
\]  

(3)

where \(rss_\omega\) and \(rss_\Omega\) are residual sum of squares under models \(\omega\) and \(\Omega\), respectively. When the null model is true, this functional \(F\) statistic is distributed like a ratio of two linear combinations of infinite independent \(\chi^2\) random variables, that is

\[
\frac{\sum_{k=1}^{\infty} \lambda_k \chi^2_{(p-q)}/(p-q)}{\sum_{k=1}^{\infty} \lambda_k \chi^2_{(n-p)}/(n-p)}
\]

where \(\lambda_1 \geq \lambda_2 \geq ... \geq 0\) are eigenvalues of the covariance function \(\gamma(s,t)\) and all the \(\chi^2\) random variables are independent of each other. This null distribution can be effectively approximated by an ordinary \(F\) distribution with degrees of freedom \(df_1 = \lambda(p - q)\) and \(df_2 = \lambda(n - p)\), where

\[
\lambda = \frac{\left(\sum_{k=1}^{\infty} \lambda_k\right)^2}{\sum_{k=1}^{\infty} \lambda_k^2}
\]

is called the degrees-of-freedom-adjustment-factor [7].

In practice, \(y_i(t)\) are assumed to be observed on evenly spaced fixed time points \(t_1, t_2, ..., t_m\) for easy interpretation and estimation. This happens in many designed experiments or studies. Then, (1) becomes
\[ y_i(t_j) = x_i^T \beta(t_j) + \varepsilon(t_j), \quad \text{for } i = 1, \ldots, n, j = 1, \ldots, m. \]

And we shall replace the integration with summation, compute

\[ rss = \sum_{i=1}^{n} \sum_{k=1}^{m} (y_i(t_k) - \hat{y}_i(t_k))^2 / m \]

and estimate the degrees-of-freedom-adjustment-factor by

\[ \hat{\lambda} = \frac{\text{trace}(\hat{\Sigma})^2}{\text{trace}(\hat{\Sigma}^2)}, \quad \text{where} \]

\[ \hat{\Sigma} = \left( \sum_{i=1}^{n} \hat{\varepsilon}_i(t_j) \hat{\varepsilon}_i(t_k) / (n - p) \right)_{m \times m} \]

is the empirical covariance matrix computed from the alternative model.

After fitting a model, it is important to identify outliers and highly influential cases. Including outliers and influential cases in the analysis may result in misleading results. Faraway [2] suggested to examine:

- Plots of the estimated eigenfunctions and their associated eigenvalues. These plots show the nature of the unexplained variation in the model.
- Normal quantile – quantile (Q-Q) plots of the estimated scores of each residual curve. These plots are useful for detecting outliers and for assessing the assumption of Gaussian error process.
- Plots of residuals versus fitted for each time point \( t_j \). In scalar regression, these plots provide useful information for checking model assumption and outliers.

However, this ordinary approach doesn’t work very well on functional data analysis, since it ignores the fact that the residuals are correlated over time \( t \). Shen and Xu
[8] developed diagnostic procedures for functional regression that consider each curve as a point in a functional space. They formally defined functional studentized residuals

$$ S_i = \frac{\sqrt{\int \hat{\varepsilon}_i^2(t) dt}}{\sqrt{(1 - h_i) \text{rss}/(n - p)}}, $$

and jackknife residual as

$$ J_i = \frac{\sqrt{\int (y_i(t) - \hat{y}_i(t))^2 dt}}{\sqrt{1 + x_i^T (X_{(i)}^T X_{(i)})^{-1} x_i \sqrt{\text{rss}_{(i)}/(n - p - 1)}}} $$

Where $h_i$ is the leverage of the $i_{th}$ curve, $X(i)$ is the $X$ matrix with the $i_{th}$ row deleted, $\hat{\varepsilon}_i^2(t)$ is the $i_{th}$ residual from the model without the $i_{th}$ curve, $\hat{y}_i(t)$ is the predicted value for curve $i$ from the model without the $i_{th}$ curve, and $\text{rss}_{(i)}$ is the residual sum of squares from the model without the $i_{th}$ curve. Define cook’s distance as

$$ D_i = \frac{\int (\hat{\beta}_{(i)}(t) - \hat{\beta}(t))^T (X^T X)(\hat{\beta}_{(i)}(t) - \hat{\beta}(t)) dt}{p \ast \text{rss}/(n - p)} $$

where $\hat{\beta}_{(i)}(t)$ is the estimate of $\beta(t)$ computed without the $i_{th}$ curve. They also proposed a Chi-square $Q-Q$ plot by plotting $S_i^2$ against quantiles of a $\chi^2$ distribution with $\hat{\lambda}$ degrees of freedom to assess the assumption of Gaussian process.

### 2.3 B-spline basis functions

B-spline basis functions can be used to smooth each $y_i(t)$ individually without reference to any particular model being fitted and can be used as preliminary step to reduce dimensionality in order to conduct further analysis. Compared with regression
analyses which typically involve the consideration of several possible models, B-spline basis functions would be advantageous. First, any particular model doesn’t have to be proposed. This avoids the problem if the form of the chosen model is incorrect. Second, after B-spline smoothing, the data intend to follow a normal distribution which is a common assumption to be made for the majority of statistical analysis.

Here, we choose smoothing splines which are equally spaced on the grid of $m$ points in $\tau$. The basis function representation is

$$y_i(t) \approx \sum_{j=1}^{m} y_{ij} B_j(t),$$

where $B_j(t), j=1, \ldots, m,$ are the basis functions. So, the function $y_i(t)$ can be reduced to a vector of length $m, y_{ij}$, for each case $i$. It is preferable to have a large $m$, particularly when $y_i(t)$ is observed at high frequency.

2.4 B-spline based multivariate likelihood ratio test (BM test)

When $y(t)$ is approximated by a vector, it is natural to look to multivariate multiple regression analysis for ideas how to test the null hypothesis that $\omega$ holds against the alternative of $\Omega$. The likelihood ratio test statistic of the null $H_{\omega}$ versus the alternative $H_{\Omega}$ is proportional to

$$\log \left| \frac{\sum_{\Omega}}{\sum_{\omega}} \right| = \sum_{j=1}^{m} \log \frac{\lambda_{\Omega}^j}{\lambda_{\omega}^j}$$
where $\lambda_j^{\Omega}$ and $\lambda_j^{\omega}$ are the decreasingly ordered eigenvalues of the empirical covariance matrices $\hat{\Sigma}_\Omega$ and $\hat{\Sigma}_\omega$, respectively. This likelihood ratio statistics depends on terms \( \log(\lambda_j^{\Omega} / \lambda_j^{\omega}) \), which do not become small as $j$ becomes large.

For functional data analysis, the dimensionality is usually much higher and dimensionality reduction techniques are required. Hence, in this study, we combine traditional multivariate log likelihood ratio test with B-spline to see if it yields a powerful overall test statistic. The process is that conducting B-spline dimension reduction, and then performing multivariate likelihood ratio test on the B-spline coefficients of this representation. This methodology has been proposed and conducted by Shen and Faraway [7].

2.5 B-spline based functional F test (BF test)

When we apply functional $F$ test directly on the functional data set, the purpose is focusing on model selection; so, the functional $F$ test doesn’t involve smoothing. An advantage of the functional $F$ test is that it is easy to compute and does not require the user to be careful about the grid size or the number of basis functions required. Meantime, a disadvantage of the functional $F$ test is that it may not be the most powerful test for comparing two nested functional linear models even though it examines important rather than trivial differences between models [7]. When we apply BM test on the functional data set, this method can be very powerful under certain circumstances. However, multivariate likelihood ratio test statistics can easily become dominated by unimportant
variations represented by the higher order eigenvectors. As pointed out by Faraway [2], traditional multivariate test statistics such as Wilks’ lambda likelihood ratio (which is a function of the likelihood ratio test statistic) are inappropriate due to the influence of unimportant variation directions. Moreover, BM test has to face the problem that how many B-spline basis functions should be selected to best represent the data.

As noted above, the dimensionality of functional data set is, often, very high and B-spline basis functions are necessary here. The smoothing technique would have some impacts on the approximation if the data for each response curve $y_i(t)$ is not quite smooth and plentiful. The data for audible noise level study is a good example of this type. Each underlying curve in the data set is considered as a linear combination of B-spline basis functions. In this study, we are interested in B-spline based functional $F$ test method (or BF test) which combines B-spline basis functions with functional $F$ test. We first represent each curve as a linear combination of $m$ B-splines and then perform the usual functional $F$ test on the coefficients of this representation. Of interest is to apply BF test and to compare with functional $F$ test applied directly on the same data set which has been done by Shen and Xu [8].

It is desirable that the BF test maintains the benefits of both functional $F$ test and BM test while at the same time limits the shortcomings of these two tests. The diagnostics procedure developed by Shen and Xu [8] can be applied directly.
CHAPTER 3

Application to the Audible Noise Data

3.1 The audible noise data

The audible noise data, one of the experiments of data reported by Nair et al. [4], is used to illustrate functional regression analysis here. This is a study to reduce audible noise levels of alternators. When an alternator rotates, a certain amount of audible noise is generated. An engineering team conducted a robust parameter design experiment to investigate the effects of 7 process assembly parameters (factors) on the audible noise levels of alternators. Table 3.1 shows the list of seven factors that were chosen for the study.

Factor D is a noise factor while all others are control factors. A 32-run experiment was conducted with a $2^{7-2}$ design with defining relation $I = CEFG = ABCDF = ABDEG$. The fractional factorial design allows estimating all main effects and control-by-noise interactions, assuming three-factor and higher-order interactions are negligible. For our purpose, the difference between control and noise factors is ignored and all seven factors are treated as the same. For each of the 32 experimental combinations, 43 measurements of sound pressure levels (responses) were recorded at rotating speeds (signal factor)
ranging from 1000 to 2500 revolution per minute. The original data include four additional replications collected at the high levels of all factors, which we do not use here.

<table>
<thead>
<tr>
<th>Factor Label</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Through Bolt Torque</td>
</tr>
<tr>
<td>B</td>
<td>Rotor Balance</td>
</tr>
<tr>
<td>C</td>
<td>Stator Varnish</td>
</tr>
<tr>
<td>D</td>
<td>Air Gap Variation</td>
</tr>
<tr>
<td>E</td>
<td>Stator Orientation</td>
</tr>
<tr>
<td>F</td>
<td>Housing Stator Slip Fit</td>
</tr>
<tr>
<td>G</td>
<td>Shaft Radial Alignment</td>
</tr>
</tbody>
</table>

Figure 3.1 shows the 32 response curves. The plots in Figure 3.1 suggest that it will be difficult to capture the effects of the design factors through a parametric model. It is, also, clear that there is no parsimonious model that would fit the data, so the functional regression approach is concerned here.

3.2 Application of B-spline based functional F test

In Shen and Xu’s study [8], functional regression analysis (i.e., functional F test) was directly applied on the audible noise data set. Their results suggested that A, C, D, and G are significant main effects and that case 16 is an outlier at the 5% level. In this study, 20-basis BF test is performed on the same data set. After 20 B-spline basis functions is applied to the audible noise data, 43 dimensions decrease to 20. Then, functional F test is applied on these 20 B-spline expansion coefficients. Table 3.2 gives
the $F$ statistics and p-values for the main effects model. The residual sum of squares is 468.32 and estimated adjustment factor $\hat{\lambda}$ is 7.05. The p-value of effect A is computed as the upper tail probability of 2.13 under an ordinary $F$ distribution with degrees of freedom 7.05 and 169.2. Note that $\hat{\lambda}(n - p) = (7.05)(32 - 8) = 169.2$. The p-values for the rest of factors can be obtained similarly. The p-values show that D and G are very significant, A and C are significant at the 5% level.

Similar to Shen and Xu’s [8], the main effects model is reduced to a simplified model with main effects A, C, D, and G. Table 3.3 shows the $F$ statistics and p-values for the reduced model. The residual sum of squares is 534.41 and estimated adjustment factor $\hat{\lambda}$ is 7.41. From the Table 3.3, it is good to know that all the four factors are still significant at the 5% level.

The reduced model is compared with the main effects model through functional $F$ test as in (3):

$$F = \frac{(\text{rss}_\omega - \text{rss}_\Omega)/(p - q)}{\text{rss}_\Omega/(n - p)} = \frac{(534.41 - 468.32)/(8 - 5)}{468.32/(32 - 8)} = 1.13$$

compared with an $F$ distribution with $df_1 = (7.05)(8 - 5) = 21.15$ and $df_2 = (7.05)(32 - 8) = 169.2$, the p-value is 0.32. So the reduced model is accepted.

The diagnostics plots for the reduced model are shown in Figure 3.2. There is no any obvious pattern in the plot of Residuals vs. Fitted. Visually, case 16 could be considered as an outlier; however, the formal $F$ test with Bonferroni adjustment does not claim that case 16 is an outlier at the 5% level. One possible explanation is that the B-spline method has a smoothing effect and filters out aberrant observations, here, case 16.
And all the points are narrow concentrated around one straight line in Chi-square \( Q-Q \) plot, illustrating that the Gaussian assumption is reasonable.

Table 3.2: \( F \) Statistics and P-values for the Main Effects Model

<table>
<thead>
<tr>
<th>Numerator</th>
<th>Denominator</th>
<th>( F ) value</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>4003.1</td>
<td>6564.6</td>
<td>0.0000</td>
</tr>
<tr>
<td>A</td>
<td>1.2965</td>
<td>2.1261</td>
<td>0.0429</td>
</tr>
<tr>
<td>B</td>
<td>0.8006</td>
<td>1.3130</td>
<td>0.2465</td>
</tr>
<tr>
<td>C</td>
<td>1.5610</td>
<td>2.5598</td>
<td>0.0154</td>
</tr>
<tr>
<td>D</td>
<td>2.5548</td>
<td>4.1896</td>
<td>0.0003</td>
</tr>
<tr>
<td>E</td>
<td>0.6102</td>
<td>1.0006</td>
<td>0.4329</td>
</tr>
<tr>
<td>F</td>
<td>0.6545</td>
<td>1.0732</td>
<td>0.3829</td>
</tr>
<tr>
<td>G</td>
<td>4.0388</td>
<td>6.6233</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Table 3.3: \( F \) Statistics and P-values for the Reduced Model

<table>
<thead>
<tr>
<th>Numerator</th>
<th>Denominator</th>
<th>( F ) value</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>4003.1</td>
<td>6471.9</td>
<td>0.0000</td>
</tr>
<tr>
<td>A</td>
<td>1.2965</td>
<td>2.0961</td>
<td>0.0421</td>
</tr>
<tr>
<td>C</td>
<td>1.5610</td>
<td>2.5237</td>
<td>0.0146</td>
</tr>
<tr>
<td>D</td>
<td>2.5548</td>
<td>4.1304</td>
<td>0.0002</td>
</tr>
<tr>
<td>G</td>
<td>4.0388</td>
<td>6.5297</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Actually, 15-basis and 25-basis BF tests are performed as well. The results are the same for three different B-spline basis functions. A, C, D, and G are all significant in the main effects model. All three \( F \) statistics show that the reduced models are accepted with these four main effects. It is of interest to note that as the number of B-spline basis functions increases, for instance, from 15-basis to 25-basis, the estimated adjustment
factors decrease from 7.612 to 7.302. Hence, it is predictable that when the number of B-spline basis functions increases to 43 which means no any B-spline smoothing techniques applied indeed, the estimated adjustment factors will decrease to 5.14 [8] and the BF test would approach functional $F$ test. This is understandable since BF test is a combination of B-spline smoothing technique and functional $F$ test.
CHAPTER 4

Simulation Study of Size and Power

4.1 Simulation study of size and power

To evaluate the size and power of the BF test together with the functional $F$ test and BM test, simulation study under similar conditions to the audible noise data is conducted. In Shen and Xu’s simulation study [8], they compared functional $F$ test with four to 10-basis BM tests and suggests that eight B-splines basis functions is the most powerful one for empirical covariance structure. Shen and Faraway [7], Shen and Xu [8], and Yang et al. [9] used Gaussian error process in their simulations. Particularly, in this thesis, it is of interest to investigate performance of BF test in comparison with functional $F$ test and BM test. In addition, whether functional $F$ test is sensitive to the departure from Gaussian random error distribution is also of interest.

The response curves are simulated using the weighted average of predicted levels from the reduced model ($\omega$, i.e., the one with four predictors) and the main effects model ($\Omega$, the one with seven predictors) plus multivariate $t$ distribution errors with different degrees of freedom from three covariance structures: compound symmetric (CS), empirical covariance (emp), and autoregressive type 1 (AR(1)). For the CS covariance
structure, the correlation coefficient between \( y_{ij} \) and \( y_{ik} \) is the same (i.e., \( \rho \) for \( j \neq k \)); for the empirical covariance structure, the covariance function \( \gamma(s,t) \) can be estimated by the empirical covariance matrix \( \hat{\Sigma} = \left( \sum_{j=1}^{n} \hat{\epsilon}_j(t_j)\hat{\epsilon}_k(t_k) / n - p \right)_{mm} \); for the AR (1) case, the correlation strength depends on the distance between the two observations (i.e., \( \rho^{j-k} \)). For the CS and AR(1) covariance structures, the residual variance is set as \( \text{var}(y_{ij}) = \sigma^2 = 10 \), a value approximated to the empirical variance computed from the reduced model. The weights are varied between 0 (corresponding to the reduced model) and 1 (corresponding to the full model) with an increment of 0.2. For each weight, 1000 sets of response curves are generated and functional \( F \) test, BF test, and BM test are applied on each simulated data set, respectively, at significance level 0.05.

Functional \( F \) test requires that the error has a Gaussian process. One of the objectives of this thesis is to see whether functional \( F \) test is still applicable when random error multivariate \( t \) distribution is assumed. The random error multivariate \( t \) distribution is generated by \( rmvt \) function in R package \( mvtnorm \) [3].

Another objective of the simulation is to investigate whether BF and BM tests are sensitive to the selection of the number of basis functions used. In the simulation, we computed BF and BM tests based on four to 30 basis functions; we reported results on five, eight, 11, and 20 basis functions for BM tests and reported results on 15, 20, 25, and 30 basis functions for BF tests.
4.2 Comparison of simulation study under compound symmetric $I$

Table 4.1 shows the powers (i.e. the probability of correctly rejecting the reduced model) of the three tests for compound symmetric covariance structure $I$ with different $t$ distribution’s degrees of freedom and correlation level at $\rho = 0.5$. It shows that BM test is always the most powerful one.

The plots in Figure 4.1 indicate the simulated power curves of the three tests under covariance structure $I$ at correlation level $\rho = 0.5$. On each plot, the $x$-axis indicates the Weights (0 ~ 1 with increment of 0.2) used for simulating 1000 data sets at each weight and $y$-axis corresponds to the power for each method on the 1000 data sets.

When weight is zero, the reduced model is the true model and the power is the size of the test. For all three methods, the nominal significance level is 0.05; but, due to various approximations, there are some errors which can be acceptable. For BF test, the simulated sizes are below the specified significance level 0.05, which means BF test has a conservative size. Table 4.2 shows the sizes of these three methods under CS covariance structure $I$ with $\rho = 0.5$.

When the weight is greater than zero, the true model is not the reduced model any more, so the probability of rejecting the reduced model becomes the power of the test. When the weight is greater than 0.3 for $df=500$ and $df=50$, 0.4 for $df=10$, and 0.5 for $df=5$, 20-basis BF test is more powerful than functional $F$ test; see Figure 4.1. It is obvious that BM test has overall significantly higher power than the other two tests. When weight is greater than or equals to 0.6, the powers of BM test equal to 1.
When the correlation level at $\rho = 0.8$ was used under compound symmetric covariance structure $\mathbf{I}$, the similar results are obtained; see Table 4.3 and 4.4. BM test is the most powerful one; BF test is the second and functional $F$ test is the least powerful one. When weight equals to 0.4, the powers of BM test equal to 1 which is too powerful; see Figure 4.2. Also, it is of interest to notice that the powers of functional $F$ test and BF test increase as degrees of freedom increase under both correlation levels at 0.5 and 0.8.

Table 4.1: Statistical Powers of the F-test, BF test, and BM test under Compound Symmetric Covariance Structure $\mathbf{I}$ with $\rho = 0.5$

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Table 4.2: Statistical Sizes of the F-test, BF test, and BM test under Compound Symmetric Covariance Structure $\mathbf{I}$ with $\rho = 0.5$

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Table 4.3: Statistical Powers of the F-test, BF test, and BM test under Compound Symmetric Covariance Structure I with $\rho = 0.8$

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Table 4.4: Statistical Sizes of the F-test, BF test, and BM test under Compound Symmetric Covariance Structure I with $\rho = 0.8$

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</tbody>
</table>
4.3 Comparison of simulation study under empirical covariance structure

Table 4.5 shows the powers of three tests for empirical covariance structure with different \( t \) distribution’s degrees of freedom, respectively. It is seen that BF test (with 20 or 25 basis) is the most powerful as the degrees of freedom are larger than 50 (included), while functional F test is the least. When degree of freedom is equal to five or ten, 8-basis BM test is the most powerful one. BF tests with 20, 25, and 30 basis functions are among the most powerful tests. On the other hand, BM tests with five and 20 basis functions have the smallest power. For the empirical covariance structure, BF test is less sensitive to the choice of the number of basis functions than the BM test.

Table 4.6 shows the sizes of these three methods. For functional \( F \) test and BF test, the simulated sizes are below the pre-specified significance level 0.05, indicating these two tests were conservative and have more space to increase their powers.

The plots in Figure 4.3 show the powers of three tests for the empirical covariance structures. When degree of freedom is larger than 50, BF test is the most powerful one. Meanwhile, it is also noticeable that BF test and BM test are comparable to each other with similar patterns in terms of power. It is of interest to notice that the powers of three tests increase as degrees of freedom increase; especially when degrees of freedom increase from 5 to 10.
Table 4.5: Statistical Powers of the F-test, BF test, and BM test under Empirical Covariance Structure II

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Table 4.6: Statistical Sizes of the F-test, BF test, and BM test under Empirical Covariance Structure II

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4.4 Comparison of simulation study under autoregressive type 1 covariance structure III

Table 4.7 shows the powers of three tests for AR (1) covariance structure with correlation level at $\rho = 0.5$. Figure 4.4 shows that the powers of BF test are lower than those of the other two tests when weights are larger than 0.2 for df=500, 50 and 10, 0.4 for df=5. It is observed from Table 4.8 that the size of F-test is much smaller than those of the other two tests. It is interesting that F-test has the largest power although it has the smallest size.

Table 4.9 shows the powers of three tests for AR (1) covariance structure with correlation level at $\rho = 0.8$. According to Table 4.9, BF test is the most powerful test although it has the smallest size (see Table 4.10). When you look at Figure 4.5, you will find out that the power curves under AR(1) error covariance structure with $\rho = 0.8$ indicate that 20-basis BF test has the largest power for this complicated error process.

Table 4.7 and 4.9 show that for the AR(1) covariance structure, BF test is less sensitive to the choice of the number of basis functions than BM test.
Table 4.7: Statistical Powers of the F-test, BF test, and BM test under Autoregressive
Type 1 Covariance Structure \( \text{III} \) with \( \rho = 0.5 \)

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Table 4.8: Statistical Sizes of the F-test, BF test, and BM test under Autoregressive Type
1 Covariance Structure \( \text{III} \) with \( \rho = 0.5 \)

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Table 4.9: Statistical Powers of the F-test, BF test, and BM test under Autoregressive Type 1 Covariance Structure $\textit{III}$ with $\rho = 0.8$

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Table 4.10: Statistical Sizes of the F-test, BF test, and BM test under Autoregressive Type 1 Covariance Structure $\textit{III}$ with $\rho = 0.8$

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4.5 Summary of simulation study

The results of simulation show that the covariance structure of the error process is influential to the power of the tests; different degrees of freedoms have effects on the power of the tests as well. When the correlation structure is autoregressive type 1, we would recommend both functional $F$ test and BF test since the simulation study has shown that these two tests have the overall largest powers among all three tests considered with respectively correlation level. Specifically, when correlation level is $\rho = 0.8$, BF test is preferable; when correlation level is $\rho = 0.5$, functional $F$ test is the most powerful test. In contrast, if the error covariance structure is like the compound symmetric type, BM test is the most powerful one. As discussed in the Chapter 2, however, the likelihood ratio test statistics may be influenced by the unimportant directions of variation and so generate statistically significant, but practically less meaningful, test results. We are also able to know that BF test has a significant larger power over functional $F$ test under compound symmetric covariance structure. In empirical covariance structure, we would like to recommend BF test to be generally conducted because it is the overall most powerful one and is not sensitive to the number of B-spline basis functions used.

In the compound symmetric covariance structure, when correlation level $\rho$ increases from 0.5 to 0.8, the powers of functional $F$ test and BF test are degrade. While in the autoregressive type 1 covariance structure, when $\rho$ increases from 0.5 to 0.8, there is no any obvious pattern for the simulated power curves we can obtain. One possible
reason is that AR(1) has more complicated covariance structure than the CS covariance structure.

For all three covariance structures, it is seen that as degrees of freedom elevate up under considered covariance structure, the powers of three tests are strengthened. In addition, the simulation study shows that BM test has the overall accurate sizes in all covariance structures.

When we did simulation study, we actually computed all tests with 4 to 30 B-spline basis functions. We only present results using certain B-spline basis functions to represent all B-spline basis functions in order to give these tests more flexibility to be applied in practice.
CHAPTER 5

Conclusions and Discussions

In this study, we propose B-spline based functional $F$ test for functional linear models and compare it with functional $F$ test and B-spline based multivariate likelihood ratio test.

The simulation study shows that which method should be adopted is determined by data set’s covariance structure. For the compound symmetric type, it is more reasonable to recommend BF test rather than BM test since BM test probably generated practically less meaningful test results due to the dominance of unimportant directions of variation. For the autoregressive type 1, we would like to recommend both $F$ test and BF test. The empirical covariance structure appears to be similar to the AR(1) structure with $\rho = 0.8$. It also indicates that Gaussian distribution for assumed error process is not crucial since our simulation study has shown that all three tests all work well even the degrees of freedom are as small as 10. Like traditional scalar $F$ test, functional $F$ test, BF test, and BM test are robust against the mild departure from Gaussian distribution (provided the error process is symmetrical as in a multivariate t distribution). The power curves of three tests in any cases have consistent relations which indicate the assumption of Gaussian stochastic process is not critical for functional data analysis to some extent.
Tables of statistical powers of three tests also show that the selection of the number of B-spline basis functions in BF test is not as critical as in BM test. B-spline basis functions selected to be applied in BF test generate consistent powers within different degrees of freedom. On the other hand, the powers of BM test are more sensitive to the selection of the number of B-spline basis functions.

According to above discussions, BF test and \( F \) test are more reliable than BM test. And they have the following advantages. First, they can be simply computed without any strong parametric assumptions; secondly, they work well when the grid sizes become large; thirdly, they are not easily influenced by unimportant variation directions. All need for functional regression analysis is that a sufficiently large amount of data should be available so that the function can be fully approximated.

However, there is still a great deal of fundamental development need to be done in the area of functional regression analysis. The inferences we draw from simulation study are tentative, and some further theoretical investigation would be valuable. Also, there are limitations with B-spline based functional \( F \) test method for functional regression analysis. First, it does not handle functional predictors, which restricts its application in some practical settings. Second, how many basis functions should be chosen to best represent the raw data is a problem. Even though we can conclude that BF test is not sensitive to the selection of the number of B-spline basis functions, we will still experience this situation more or less.
Figure 3.1: Audible Noise Observed Response Curves
Figure 3.1(Continued): Audible Noise Observed Response Curves
Figure 3.2: Diagnostics Plots for the Reduced Model
Figure 4.1: Simulated Power Curves under Compound Symmetric Covariance Structure

with $\rho = 0.5$
Figure 4.2: Simulated Power Curves under Compound Symmetric Covariance Structure with $\rho = 0.8$. 
Figure 4.3: Simulated Power Curves under Empirical Covariance Structure
Figure 4.4: Simulated Power Curves under Autoregressive Type 1 Covariance Structure

with $\rho = 0.5$. 
Figure 4.5: Simulated Power Curves under Autoregressive Type 1 Covariance Structure with $\rho = 0.8$. 
REFERENCES


